10.2 Newton's Method for Systems

Problem: we want to find the roots of the nonlinear system

$$\begin{cases} x_1^2 - 10x_1 + x_2^2 + 8 = 0 \\ x_1 x_2^2 + x_1 - 10x_2 + 8 = 0 \end{cases}$$

Derivation of Newton Method for Systems.

Taylor's thm for functions of 2 Variables:

$$f : D \subset \mathbb{R}^2 \longrightarrow \mathbb{R}$$
$$(x_1, x_2) \longrightarrow f(x_1, x_2)$$

Let $f \in C^4(D)$ (4 Times cont. differentiable) and $(p_1, p_2) \in D$
Then, f can be represented by a Taylor polynomial in the variables x_1, x_2, given by

$$f(x_1,x_2) = f(\underbrace{p_1, p_2}_{=\vec{p}}) + f_{x_1}(\vec{p})(x_1 - p_1) + f_{x_2}(\vec{p})(x_2 - p_2)$$
$$+ f_{x_1x_1}(\vec{p})\frac{(x_1-p_1)^2}{2} + f_{x_1x_2}(\vec{p})(x_1-p_1)(x_2-p_2)$$
$$+ f_{x_2x_2}(\vec{p})\frac{(x_2-p_2)^2}{2}$$
$$+ \frac{1}{3!}\sum_{i,j,k=1}^{2} f_{x_ix_jx_k}(\vec{p})(x_i-p_i)(x_j-p_j)(x_k-p_k)$$
$$+ R_n(\vec{x}(\xi)) \text{ Remainder.}$$

In brief,

$$f(\vec{x}) = f(\vec{p}) + \nabla f(\vec{p}) \cdot (\vec{x} - \vec{p}) + \cdots$$

Extension to Vector functions in two Variables:

If
$$F: D \subset \mathbb{R}^2 \longrightarrow \mathbb{R}^2$$
$$(x_1, x_2) \longrightarrow \vec{F}(x_1, x_2) = \begin{pmatrix} f_1(x_1, x_2) \\ f_2(x_1, x_2) \end{pmatrix}$$

$\vec{F}$ Suff. differentiable on D

and $\vec{p} \in D$, then for any $\vec{x} \in D$

$$\vec{F}(x_1, x_2) \simeq \vec{F}(p_1, p_2) + \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \partial f_2/\partial x_1 & \partial f_2/\partial x_2 \end{bmatrix} (\vec{p}) \begin{bmatrix} x_1 - p_1 \\ x_2 - p_2 \end{bmatrix} + O(\|\vec{x} - \vec{p}\|^2)$$

then, $\vec{F}(x_1, x_2) = \vec{0}$ or (x_1, x_2) is a root of $\vec{F}$

if
$$\vec{0} \simeq \vec{F}(\vec{p}) + J(\vec{p})(\vec{x} - \vec{p}).$$

and a root of $\vec{F}$ can be approximated by $\vec{x}$ satisfying

$$(\vec{x} - \vec{p}) = -J^{-1}(\vec{p})\,\vec{F}(\vec{p})$$

or
$$\boxed{\vec{x} = \vec{p} - J^{-1}(\vec{p})\,\vec{F}(\vec{p})}$$

Functional Iteration:

$$\boxed{\vec{p}_{n+1} = \vec{p}_n - J^{-1}(\vec{p}_n)\vec{F}(\vec{p}_n)} \quad (3.1)$$

It requires $J(\vec{p})$ to be invertible.

The computation is performed in two steps without computing $J^{-1}(\vec{p})$. In fact,

Calling $\boxed{\vec{y}_{n+1} = \vec{p}_{n+1} - \vec{p}_n} \quad (3.2)$

then

$$\vec{y}_{n+1} = -J^{-1}(\vec{p}_n)\vec{F}(\vec{p}_n)$$

$$\Leftrightarrow \quad \boxed{J(\vec{p}_n)\vec{y}_{n+1} = -F(\vec{p}_n)}$$

We solve this system for $\vec{y}_{n+1}$. and as a next step, we solve for $\vec{p}_{n+1}$ from (3.2)

$$\boxed{\vec{p}_{n+1} = \vec{y}_{n+1} + \vec{p}_n}. \quad (3.3).$$

In MATLAB sintaxis:

$$\vec{p}_{n+1} = \vec{p}_n - J(\vec{p}_n)\backslash F(\vec{p}_n)$$

Nonlinear Systems Newton Method.

Exercise #3 Sect. 10.1

$$\begin{cases} \underbrace{x_1^2 - 10x_1 + x_2^2 + 8}_{=f_1(x_1,x_2)} = 0 \\ \underbrace{x_1 x_2^2 + x_1 - 10x_2 + 8}_{=f_2(x_1,x_2)} = 0 \end{cases}$$

(1,1) is a root.

$$\vec{F}(\vec{x}) = \vec{F}(x_1,x_2) = (f_1(x_1,x_2),\ f_2(x_1,x_2)) = \vec{0}$$

Newt. method

$$\vec{G}(\vec{x}) = \vec{x} - J^{-1}(\vec{x})\vec{F}(\vec{x}) \qquad \vec{x}_n = \vec{x}_{n-1} - J^{-1}(\vec{x}_{n-1})F(\vec{x}_{n-1})$$

or $\vec{x}_n - \vec{x}_{n-1} = -J^{-1}(\vec{x}_{n-1})F(\vec{x}_{n-1})$

or $\vec{y}_n = -J^{-1}(\,)F(\,)$

or

$$\boxed{\begin{aligned} J(\vec{x}_{n-1})\vec{y}_n &= -F(\vec{x}_{n-1}) \\ \vec{y}_n &= \vec{x}_n - \vec{x}_{n-1}. \end{aligned}}$$

In our example,

$$J(\vec{x}) = \begin{bmatrix} \frac{\partial f_1}{\partial x_1}(\vec{x}) & \frac{\partial f_1}{\partial x_2}(\vec{x}) \\ \frac{\partial f_2}{\partial x_1}(\vec{x}) & \frac{\partial f_2}{\partial x_2}(\vec{x}) \end{bmatrix} = \begin{bmatrix} 2x_1 - 10 & 2x_2 \\ x_2^2 + 1 & 2x_1x_2 - 10 \end{bmatrix}$$

Functional Iteration:

Initial Guess: $\vec{x}^0 = (x_1^0, x_2^0) \overset{\text{for example}}{=} (1/2, 1/2)$

1st step: $\boxed{J(\vec{x}^0)\,\vec{y}^{(0)} = -\vec{F}(\vec{x}^0)}$

$$\begin{bmatrix} 2x_1^0 - 10 & 2x_2^0 \\ (x_2^0)^2 + 1 & 2x_1^0x_2^0 - 10 \end{bmatrix} \begin{bmatrix} y_1^0 \\ y_2^0 \end{bmatrix} = - \begin{bmatrix} (x_1^0)^2 - 10x_1^0 + (x_2^0)^2 + 8 \\ x_1^0(x_2^0)^2 + x_1^0 - 10x_2^0 + 8 \end{bmatrix}$$

Solve this linear system to obtain $\vec{y}^{(0)} = \begin{pmatrix} y_1^0 \\ y_2^0 \end{pmatrix}$.

<u>2nd step:</u> $\vec{X}^{(1)} = \vec{X}^{(0)} + \vec{y}^{(0)}$

An updated approx. $\vec{x}^{(1)}$ to the root $\vec{x}^*$ has been obtained.

<u>3rd step:</u> check tolerance satisfaction.

$$\| \vec{X}^{(1)} - \vec{X}^{(0)} \| = \left\| \begin{pmatrix} x_i^1 - x_i^0 \\ y_i^1 - y_i^0 \end{pmatrix} \right\| \le TOL.$$

or simply $\| \vec{y}^{(0)} \| \le TOL$

If yes. stop.

else. $\vec{X}^0 = \vec{X}^{(1)}$.

repeat steps 1-3.